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Convolution of Generalized Legendre Transformable functions

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Abstract: In this paper first we introduce how the Laplace equation reduces to the Legendre equation and hence the appearance of Legendre polynomial as its solution and then a beautiful result known as addition theorem has been used to determine the product of two Legendre polynomials in terms of another Legendre polynomial finally Convolution theorem established by Churchill and Dolph [2] has been extended for generalized Legendre transformable functions.

1. Introduction

In this paper we have explored the theory of extension of the classical Legendre transformation to generalized functions. At first the important properties of Legendre functions have been written. Applications of Legendre functions and Legendre transforms in physical problems have been shown. Space to which Legendre function belongs has been specified. Generalized Legendre transformation as a member of dual of that space has been defined.

2. Preliminaries.

Perhaps the easiest way to introduce the Legendre polynomial $P_n(x)$ is through its generating relation. In words we can say that when we expand the function $(1-2xt+t^2)^{-1/2}$ as the power series in t, the coefficient of t^n is a polynomial in x which is known as Legendre polynomial $P_n(x)$. Thus we have

$$(1-2xt+t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x)t^n \,. \qquad (2.1)$$

Using the binomial expansion of $(1-2xt+t^2)^{-1/2}=\{1-t(2x-t)\}^{-1/2}$ twice and on comparing the coefficients of t^n we arrive at

$$P_n(x) = \sum_{k=0}^{[n/2]} \frac{(-1)^k (1/2)_{n-k} (2x)^{n-2k}}{k!(n-2k)!}. \tag{2.2}$$

Here by [n/2] we mean intergral part of n/2 i.e.

$$[n/2] = \begin{cases} n/2, if\ n\ is\ even \\ (n-1)/2, if\ n\ is\ odd \end{cases}$$

also by $(a)_n$ we mean the product a(a+1)(a+2)…(a+n-1). For ready reference it is relevant to write below the important and useful properties of $P_n(x)$:

(i) $y=P_n(x)$ satisfies the second order differential equation

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0. \tag{2.3}$$

It can also be written in the form

$$\frac{d}{dx}\{(x^2-1)\frac{dy}{dx}\} = \mathrm{n}(\mathrm{n}+1)\mathrm{y}.$$

This means that $P_n(x)$ is an eigen function of the operator

$$\eta \equiv \frac{d}{dx}\{(x^2-1)\frac{d}{dx}\}, \tag{2.4}$$

corresponding to the eigen value n(n+1).

(ii) The sequence of polynomials $P_n(x)$, n=0,1,2… forms an orthogonal set on the interval $-1<x<1$, i.e.

$$\int_{-1}^{1} P_n(x)P_m(x)dx = 0,\ if\ \ m \neq n. \tag{2.5}$$

We also have

$$\int_{-1}^{1} P_n^2(x)dx = \frac{2}{2n+1} \tag{2.6}$$

This later equation shows that $P_n(x)$ belongs to the Hilbert space $L_2(-1,1)$. Being a polynomial it is also infinitely differentiable i.e. $P_n(x) \in C^{\infty}(-1,1)$.

(iii) It is also useful to know that the zeros of $P_n(x)$ are distinct, and all lie in the open interval (-1,1).

(iv) A most important concept in our context is the following expansion theorem Rainville[77]; "If on -1≤ x ≤1 f(x) is continuous except for a finit number of discontinuities, if on -1≤ x ≤1 f'(x) exists where f(x) is continuous and the right hand and left hand derivatives of f(x) exist at the discontinuities, and if

$$a_n = (n+\frac{1}{2})\int_{-1}^{1} f(y)P_n(y)dy, \qquad (2.7)$$

then

$$\sum_{n=0}^{\infty} a_n P_n(x) = f(x), \qquad -1 < x < 1, \qquad (2.8)$$

at the points of continuity of f(x) and we also have

$$\sum_{n=0}^{\infty} a_n P_n(x) = \frac{1}{2}[f(x+0)+f(x-0)],$$

at the points of discontinuity of f(x)

(v) For non-negative integer n, we have

$$x^n = \frac{n!}{2^n}\sum_{k=0}^{[n/2]} \frac{(2n-4k+1)}{k!(3/2)_{n-k}} P_{n-2k}(x). \qquad (2.9)$$

i.e. x^n can be expressed as a linear combination of Legendre polynomials.

(vi) It is interesting to observe that

$$P_n(x) = \frac{1}{2^n n!} D^n (x^2-1)^n. \qquad (2.10)$$

This relation is known as *Rodrigue's formula* for Legendre polynimial. Using this we obtain

$$P_n(x) = \sum_{k=0}^{n} ({}^nC_k)^2 \left(\frac{x-1}{2}\right)^{n-k} \left(\frac{x+1}{2}\right)^k. \qquad (2.11)$$

(vii) Using the generating relation (2.1) we obtain the following expression for $P_n(x)$ as a terminating hypergeometric function

$$P_n(x) = {}_2F_1\left(-n, n+1; 1; \frac{1-x}{2}\right). \qquad (2.12)$$

(viii) In an attempt to express $P_n(x)$ in integral form we arrive at the Laplace's first integral as

$$P_n(x) = \frac{1}{\pi}\int_0^{\pi}\left[x + (x^2-1)^{1/2}\cos\phi\right]^n d\phi. \qquad (2.13)$$

(ix) Using (2.13) we derive that

$$P_n(-x) = (-1)^n P_n(x), \qquad (2.14)$$

which shows that for $P_{2n}(x)$ is an even and $P_{2n+1}(x)$ is an odd function. Putting x=1 in (2.1) we get

$$(1-t)^{-1} = \sum_{n=0}^{\infty} P_n(1)t^n$$

and hence

$$P_n(1)=1$$

and from (2.14)

$$P_n(-1)=(-1)^n$$

Using (2.3), we prove that for $-1 < x < 1$,

$$|P_n(x)| < 1 \qquad (2.15)$$

and also that

$$|P_n(x)| < \left[\frac{\pi}{2n(1-x^2)}\right]^{1/2} \qquad (2.16)$$

(x) Applications of Legendre Polynomials

Many problems of physics reduce to the solution of Legendre equation (2.3). We know that the Laplace's equation in three dimensional cartesian system of coordinates is

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} = 0, \qquad (2.17)$$

which occurs very frequently in physical problems. To transform this equation in spherical polar coordinated (r, θ, ϕ) where x=rsinθ cosϕ,

y= rsinθ sinϕ, z=rcosθ if we put first x=ucosϕ, y=usinϕ where u=rsinθ in (2.17) we get ,

$$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}=\frac{\partial^2 v}{\partial z^2}+\frac{\partial^2 v}{\partial u^2}+\frac{1}{u}\frac{\partial v}{\partial u}+\frac{1}{u^2}\frac{\partial^2 v}{\partial^2 \phi^2}=0 \qquad (2.18)$$

Now putting z=rcosθ and u= rsinθ in (2.18) we get

$$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}=\frac{\partial^2 v}{\partial r^2}+\frac{2}{r}\frac{\partial v}{\partial r}+\frac{1}{r^2}\frac{\partial^2 v}{\partial^2 \theta}+$$

$$+\frac{\cot\theta}{r^2}\frac{\partial v}{\partial \theta}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 v}{\partial \phi^2}=0 \qquad (2.19)$$

Now if we put v = $r^n S_n(\theta,\phi)$ into (2.19) we get

$$\frac{\partial^2 S_n}{\partial \theta^2}+\cot\theta\frac{\partial S_n}{\partial \theta}+\operatorname{cosec}^2\theta\frac{\partial^2 S_n}{\partial \phi^2}+n(n+1)S_n=0 \qquad (2.20)$$

It should be noted here that $S_n(\theta,\phi)$ is called a *spherical surface_harmonic* of degree n.

Again if we put

$S_n(\theta,\phi) = f(\theta)\cos m(\phi+\varepsilon)$,

we get

$$\frac{d^2 f(\theta)}{d\theta^2}+\cot\theta\frac{df(\theta)}{d\theta}+\{n(n+1)-m^2\operatorname{cosec}^2\theta)\}f(\theta)=0. \qquad (2.21)$$

Now if we transform (2.21) by putting x = cosθ and y = f(θ) we get

$$(1-x^2)\frac{d^2 y}{dx^2}-2x\frac{dy}{dx}+\{n(n+1)-\frac{m^2}{1-x^2}\}y=0 \qquad (2.22)$$

The equation (2.22) is known as the *associated Legendre equation*. It is clear that for m = 0 the equation (2.22) reduces to (2.3). In this way we have described here the reduction of Laplace's equation to Legendre equation. The case m = 0 implies that v is independent of ϕ, it means that (2.19) reduces to

$$\frac{\partial^2 v}{\partial r^2}+\frac{2}{r}\frac{\partial v}{\partial r}+\frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2}+\frac{\cot\theta}{r^2}\frac{\partial v}{\partial \theta}=0 \qquad (2.23)$$

whose one solution may be written as

$$v(r,\theta) = r^n P_n(\cos\ \theta) \qquad (2.24)$$

It is also fruitful to note that one solution of

$$\left(\frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}+\frac{\cot\theta}{r^2}\frac{\partial}{\partial \theta}\right)\times$$

$$\left(\frac{\partial^2 v}{\partial v^2}+\frac{2}{r}\frac{\partial v}{\partial r}+\frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2}+\frac{\cot\theta}{r^2}\frac{\partial v}{\partial \theta}\right)=0 \qquad (2.25)$$

is

$$v(r,\theta) = r^{n+2} P_n(\cos\ \theta) \qquad (2.26)$$

The equation (2.25) is known as *biharmonic equation* in r, θ plane. It appears in the *theory of elasticity*, its solution gives *stress function*, an important quantity in this branch of physics.

(xi) A deduction from Addition Theorem for Legendre Polynomials

From Sneddon [84, Page 87] we have

$$P_n(\cos\mu)=P_n(\cos\theta)P_n(\cos\theta')$$

$$+2\sum_{m=1}^{n}(-1)^m\frac{(n-m)!}{(n+m)!}P_n^m(\cos\theta)P_n^m(\cos\theta')\cos m\phi \qquad (2.27)$$

where

$$\cos\mu=\cos\theta\cos\theta'+\sin\theta\sin\theta'\cos\phi, \qquad (2.28)$$

and $P_n^m(\cos\theta)$ is associated Legendre polynomial. Integrating (2.27) with respect to ϕ from 0 to π we get

$$P_n(\cos\theta)P_n(\cos\theta')=\frac{1}{\pi}\int_0^{\pi} P_n(\cos\mu)\,d\phi \qquad (2.29)$$

If we put cos θ=x and cos θ′=y, the we get

$$P_n(x)P_n(y)=\frac{1}{\pi}\int_0^{\pi} P_n(xy+\sqrt{(1-x^2)(1-y^2)}\cos\phi\, d\phi. \qquad (2.30)$$

On putting cos ϕ = u we find that

$$P_n(x)P_n(y)=\frac{1}{\pi}\int_{-1}^{1} P_n(xy+\sqrt{(1-x^2)(1-y^2)}u)\frac{du}{\sqrt{(1-u^2)}} \qquad (2.31)$$

If we put $xy+\sqrt{(1-x^2)(1-y^2)}u)=z$ the later relation gives

$$P_n(x)P_n(y)=\frac{1}{\pi}\int_{xy-\sqrt{(1-x^2)(1-y^2)}}^{xy+\sqrt{(1-x^2)(1-y^2)}} P_n(z)(1-x^2-y^2-z^2+2xyz)^{(-1/2)}dz \qquad (2.32)$$

(xii) A Series of Products of Three Legendre Polynomials

J.P.Vinti [97] has proved that.

If x,y and z are real variables and P_n is Legendre Polynomial of order n then,

$$\sum_{n=o}^{\infty}(n+\frac{1}{2})P_n(x)P_n(y)P_n(z)=\frac{1}{\pi}(1-x^2-y^2-z^2+2xyz)^{-1/2} \qquad (2.33)$$

if $1-x^2-y^2-z^2+2xyz>0$

and this sum is zero if

$$1-x^2-y^2-z^2+2xyz<0$$

3. The Generalized Legendre Transformation.

Following Zemanian [105] we take the interval I = (-1,1), the differential operator

$$\eta = \frac{d}{dx}\{(x^2 - 1)\frac{d}{dx}\}$$

(mentioned in 2.4). As usual let $L_2(I)$ denote the space of all quadratically intergrable functions defined on I. This means that if $f \in L_2(I)$ then f is complex valued function defined on (-1,1) and is such that

$$\alpha_0(f) = [\int_{-1}^{1} |f(x)|^2 dx]^{1/2} < \infty. \tag{3.1}$$

If f, $g \in L_2(I)$ and are such that

$$\alpha_0(f - g) = [\int_{-1}^{1} |f(x) - g(x)|^2 dx]^{1/2} = 0 \tag{3.2}$$

Then we say that f and g are equivalent functions. In this case we write $f \sim g$. Clearly it is an equivalence relation, hence $L_2(I)$ is divided into equivalence classes. Infact two functions in the same equivalence class may have different values at subset J of I whose Lebesque measure is zero. In other words we say that f and g are equal almost everywhere.

Having $L_2(I)$ as a family of equivalence classes in our mind we define sum of functions (in different classes) and scalar multiple of a function in $L_2(I)$. It is known that $L_2(I)$ satisfies all the conditions of a *vector space*. This vector space is made an inner product space by defining *inner product* of f, $g \in L_2(I)$ by the relation

$$(f, g) = \int_{-1}^{1} f(x)\overline{g(x)}\, dx \tag{3.3}$$

We can verify the conditions of inner product.

It is clear that

$$(f, f) = \int_{-1}^{1} |f(x)\overline{f(x)}\, dx = \int_{-1}^{1} |f(x)|^2 dx$$

$$= [\alpha_0(f)] \geq 0 \tag{3.4}$$

Hence in this way the vector space $L_2(I)$ has been made into a *Hilbert space*, the norm in it is defined (through inner product) as

$$\| f \| = \alpha_0(f) = \{(f,f)\}^{1/2} = \left(\int_{-1}^{1} | f(x) |^2 \, dx \right)^{1/2} \tag{3.5}$$

The mertic in $L_2(I)$ is defined as

$$d(f,g) = \alpha_0(f-g) = \left(\int_{-1}^{1} | f(x) - g(x) |^2 \, dx \right)^{1/2} \tag{3.6}$$

It is known that $L_2(I)$ turns out to be complete as a metric space i.e. every Cauchy sequence in it has a limit in itself.

The sequence $\{P_n(x)\}$, n=0,1,2,….. is a complete set of orthogonal polynomials in this Hilbert space $L_2(I)$ hence using (2.5) and (2.6) we can say that the sequence $\{\psi_n(x)\} = \left\{ \sqrt{(n+\frac{1}{2})}\, P_n(x) \right\}$ is complete orthonormal system in $L_2(I)$. Due to a well known theorem in Hilbert space we get that every arbitrary element $f \in L_2(I)$ has Fourier Legendre expansion in the form

$$f(x) = \sum_{n=o}^{\infty} (n+\frac{1}{2})(f(t), P_n(t)) P_n(x) \tag{3.7}$$

where

$$(f(t), P_n(t)) = \int_{-1}^{1} f(t) P_n(t) \, dt$$

It is known that the complex numbers (f, P_n) called the *Fourier Legendre coefficients*.

We now select a subset of $L_2(I)$ which serves as a testing function space for the space of generalized functions whose generalized Legendre transformations may be defined. Let $\phi(x) \in L_2(I)$ be such that (i) $D^n \phi(x)$ is continuous for every n=0,1,2,….; i.e. ϕ is smooth on I (ii) for each k=0,1,2,…

$$\alpha_k \phi = \alpha_0(\eta^k \phi) = \left[\int_{-1}^{1} | \eta^k \phi(x) |^2 \, dx \right]^{1/2} < \infty \tag{3.8}$$

and

(iii) for each n, k as above, we have

$$(\eta^k \phi, \psi_n) = (\phi, \eta^k \psi_n) \tag{3.9}$$

Let us denote this subset of $L_2(I)$ as $A(P_n)$. The sum of any two members of $A(P_n)$ is a member of $A(P_n)$ and scalar multiple of a member of $A(P_n)$ is again a member of $A(P_n)$. We can verify that $A(P_n)$ satisfies all the properties of a vector space. This vector space is made into a topolyical vector space by defining topology generated by separating collection of seminorms α_k, k=0,1,2… defined by (8). This topology having a countable local base is metrizable. The suitable metric is defined by

$$d(\phi,\psi)=\sum_{k=0}^{\infty}\frac{2^{-k}\alpha_k(\phi-\psi)}{1+\alpha_k(\phi-\psi)} \tag{3.10}$$

where ϕ, $\psi \in A(P_n)$. Following Rudin [81] and Zemanian [102] it is clear that d is complete hence $A(P_n)$ is a *Frechet space*. We define the Legendre transform of $\phi \in A(P_n)$ by the relation

$$T(\phi)=\Phi(n)=\int_{-1}^{1}\phi(x)\bar{\psi}_n(x)dx=(\phi,\psi_n) \tag{3.11}$$

where $\psi_n(x)=\sqrt{(n+1/2)}\;P_n(x)$ is normalized Legendre polynomial. Here we clarify that;

$$T[\eta(\phi]=\int_{-1}^{1}\{\frac{d}{dx}(-1+x^2)\frac{d}{dx}\}(\phi)\bar{\psi}_n(x)dx$$

$$=\int_{-1}^{1}\{\frac{d}{dx}(-1+x^2)\frac{d\phi}{dx}\}\sqrt{(n+1/2)}\,P_n(x)dx$$

$$=\{(-1+x^2)\frac{d\phi}{dx}\sqrt{(n+1/2)}\;P_n(x)\}_{-1}^{1}$$

$$=-\int_{-1}^{1}(-1+x^2)\frac{d\phi}{dx}\sqrt{(n+1/2)}\frac{d}{dx}P_n(x)dx$$

$$=-\int_{-1}^{1}\frac{d\phi}{dx}\sqrt{(n+1/2)}(-1+x^2)\frac{d}{dx}\bar{P}_n(x)dx$$

$$=-\left[\{\phi(x)\sqrt{(n+1/2)}(-1+x^2)\frac{d}{dx}\bar{P}_n(x)\}_{-1}^{1}\right.$$

$$-\int_{-1}^{1} \phi(x)\sqrt{(n+1/2)}\,\frac{d}{dx}\{(-1+x^2)\frac{d\bar{P}_n(x)}{dx}\}dx \Bigg]$$

$$= n(n+1)\int_{-1}^{1} \phi(x)\ \bar{\psi}_n(x)dx = n(n+1)(\phi,\psi_n)$$

$$= n(n+1)T[\phi]$$

We may also write the above relation as

$$(\eta(\phi),\psi_n) = (\phi,\eta(\psi_n)) \qquad (3.12)$$

Repeating the process we arrive at

$$(\eta^k(\phi),\psi_n) = (\phi,\eta^k(\psi_n)) \qquad (3.13)$$

thus (9) is satisfied in our case.

Churchill[16] derived certain operational properties of Legendre transformation of certain members of A(P_n). The first one is derived from (12) i.e. from

$$(\eta(\phi),\psi_n) = (\phi, n(n+1)\psi_n)$$

$$= (\phi,\{(n+\frac{1}{2})^2 - \frac{1}{4}\}\psi_n)$$

$$= (n+\frac{1}{2})^2(\phi,\psi_n) - \frac{1}{4}(\phi,\psi_n) \qquad (3.14)$$

It should also be kept in mind that when (ϕ,ψ_n) denotes the Legendre transform of ϕ, the relation (7) i.e. for $\phi \in$ A(P_n) we have

$$\phi = \sum_{n=0}^{\infty} (\phi,\psi_n)\psi_n, \qquad (3.15)$$

gives the process of getting ϕ from the sequence (ϕ,ψ_n) of numbers. Hence (3.15) is inverse of Legendre transformation. More explicitly we can write

$$\phi(x)=\sum_{n=0}^{\infty}\Phi(n)\psi_n(x)=T^{-1}(\Phi(n)) \qquad (3.16)$$

where $\Phi(n)=(\phi,\psi_n)$

4. Convolution:

One of the important problem in the theory of integral transformation is that, knowing the transforms of two given functions F(x) and G(x) namely f(n) and g(n) respectively, to know the inverse transform of the product f(n).g(n). This also means that finding a function H(x) in terms of F(x) and G(x) who transform is f(n).g(n). The result concerning such problem is know as *convolution*.

Churchill and Dolph [1] have obtained the convolution of two Legendre transformable functions by two different methods. In [1] they have obtained the convolution by finding the inverse transform of the product of transforms and in [2] by writing the product of transforms as transform of the function which turns out to the convolution.

By applying suitable transformations of variables they have written the convolution in different forms given below. The convolution H of F and G may be written as;

$$H(\cos\mu)=\frac{1}{\pi}\int_0^{\pi} F(\cos\lambda)\sin\lambda\, d\lambda \quad \times$$

$$\int_0^{\pi} G(\cos\lambda\cos\mu+\sin\lambda\sin\cos\alpha)d\alpha, \qquad (4.1)$$

or as

$$H(\cos\mu)=\frac{1}{\pi}\int_0^{\pi/2}\sin\phi\, d\phi\int_0^{2\pi} F[\sin\phi\sin(\beta-\frac{\mu}{2})]\times$$

$$G[\sin\phi\sin(\beta+\frac{\mu}{2})]d\beta, \qquad (4.2)$$

or as

$$H(x)=\frac{1}{\pi}\int\int_{E(x)} F(y)G(z)(1-x^2-y^2-z^2+2xyz)^{-1/2}dydz \quad (4.3)$$

where E(x) is the interior of the ellipse

$$y^2 + z^2 - 2xyz = 1-x^2 \qquad (4.4)$$

for each fixed x such that $-1 < x < 1$

The ellipse (4.4) may be written in the from

$$\frac{(z+y)^2}{2(1+x)} + \frac{(z-y)^2}{2(1-x)} = 1$$

the formula (4.3) can also be written in the form

$$H(x) = \frac{1}{\pi}\int_{-1}^{1} F(y)dy \int_{xy-\sqrt{(1-x^2)(1-y^2)}}^{xy+\sqrt{(1-x^2)(1-y^2)}} G(z)(1-x^2-y^2-z^2+2xyz)^{-1/2} dz$$

(4.5)

In paper [1] they have derived the form (4.3) by finding the inverse transform of the product of transforms by using a sum of the series in product of three Legendre functions obtained by Vinti [3]. This process may be applied to obtain the convolution of Legendre Transformable generalized functions as shown below. Let

$a_n=(F(y),P_n(y))$,

$b_n=(G(y),P_n(z))$,

then we have

$$\sum_{n=0}^{\infty} (n+\frac{1}{2})a_n b_n P_n(x)$$

$$= \sum_{n=0}^{\infty} (n+\frac{1}{2})(F(y),P_n(y))(G(z),P_n(z))P_n(x)$$

$$= \sum_{n=0}^{\infty} (n+\frac{1}{2})(F(y)G(y),P_n(y)P_n(z))P_n(x)$$

$$= \sum_{n=0}^{\infty} (n+\frac{1}{2})(F(y)G(z),P_n(x)P_n(y)P_n(z))$$

$$= (F(y)G(z),\sum_{n=0}^{\infty} (n+\frac{1}{2})P_n(x)P_n(y)P_n(z))$$

$$= (F(y)G(z), \frac{1}{\pi}(1 - x^2 - y^2 - z^2 + 2xyz)^{-1/2}), \qquad (4.6)$$

by the use of (2.32). Hence the convolution may be obtained by the action of the direct product of generalized functions F and G to

$$\frac{1}{\pi}(1 - x^2 - y^2 - z^2 + 2xyz)^{-1/2}.$$

Infact (4.6) may be thought as the analytic continuation of (4.3). The above result is formally and we have to work further for precise justification of the process.

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